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Exact NSE solutions with crystallographic symmetries and no transfer of energy through the spectrum

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Abstract

Exact space periodic solutions to the Navier–Stokes equations are derived that are invariant with respect to any crystallographic group in $\mathbb{R}^3$ with purely rotational point group Γ. A complete classification of the space periodic NSE solutions with pairwise non-interacting Fourier modes is obtained.

Keywords: Space periodic NSE solutions; Crystallographic groups; Fourier modes; Beltrami flows

1. Introduction

In this paper, we solve the problem of complete classification of solutions to the 2-D and 3-D Navier–Stokes equations

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{V} + \frac{1}{\rho}\mathbf{f}, \qquad \operatorname{div}\mathbf{V} = 0, \tag{1.1}$$

with non-interacting Fourier modes, provided that solutions are space periodic with arbitrary vector periods $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$. Here $\mathbf{V}(t, \mathbf{x})$ is the fluid velocity vector field, $p(t, \mathbf{x})$ is the pressure

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and $\mathbf{f}(t, \mathbf{x})$ is the body force, vector $\mathbf{x} = (x_1, x_2, x_3)$. We show that there are exactly four infinite families of solutions with pairwise non-interacting modes. Among them there are two classes of classically known exact solutions [1] and two classes of new exact solutions [2–5]. The most important solutions depend on all four variables t, x_1, x_2, x_3 and have no geometrical symmetries. The solutions correspond to the infinite series of invariant submanifolds for the Navier–Stokes equations.

The standard 2π-periodic solutions to the NSE were considered in papers [6–9]. We show that the Navier–Stokes dynamical systems for space periodic solutions with different vector periods $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ generically are not equivalent to each other and have qualitatively different classes of exact solutions. The moduli space of non-equivalent Navier–Stokes dynamical systems has dimension 6.

We derive exact solutions to the Navier–Stokes equations that are invariant with respect to certain crystallographic groups in $\mathbb{R}^3$. Since the general problem of existence and smoothness of solutions to the NSE is unresolved [10], we construct solutions with crystallographic symmetries in exact form. As is known [11], there are 219 non-isomorphic crystallographic groups G in three dimensions. Among them, there are 52 groups G which have purely rotational point groups $\Gamma \subset \mathrm{SO}(3)$ that can be either the octahedral group O, or tetrahedral T, or dihedral D_n, or a cyclic group C_n where $n = 2, 3, 4, 6$. We construct exact NSE solutions that are invariant with respect to any of the 52 crystallographic groups with purely rotational point groups Γ.

2. Dynamical system for the space periodic solutions

(I) We study NSE solutions that satisfy the periodicity conditions

$$\mathbf{V}(t, \mathbf{x} + \mathbf{p}_j) = \mathbf{V}(t, \mathbf{x}), \qquad \nabla p(t, \mathbf{x} + \mathbf{p}_j) = \nabla p(t, \mathbf{x}),$$
$$\mathbf{f}(t, \mathbf{x} + \mathbf{p}_j) = \mathbf{f}(t, \mathbf{x}), \tag{2.1}$$

where $i = 1, 2, 3$ and vector periods $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are linearly independent. The integral combinations $n_1\mathbf{p} + n_2\mathbf{p}_2 + n_3\mathbf{p}_3$, $n_i \in \mathbb{Z}$, form a lattice of periods Λ. Let vectors $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ are defined by the equations

$$\mathbf{k}_i \cdot \mathbf{p}_j = 2\pi\delta_{ij}. \tag{2.2}$$

The integral combinations $m_1\mathbf{k} + m_2\mathbf{k}_2 + m_3\mathbf{k}_3$, $m_i \in \mathbb{Z}$, form the reciprocal lattice Λ^*. Vectors $\mathbf{k}_i$ are $\mathbf{k}_i = \lambda\mathbf{p}_j \times \mathbf{p}_k$, where $\lambda = 2\pi[\mathbf{p}_1 \cdot (\mathbf{p}_2 \times \mathbf{p}_3)]^{-1}$ and indices i, j, k form a cyclic permutation. We present the space periodic solutions (2.1) in the form of Fourier series

$$\mathbf{V}(t, \mathbf{x}) = \sum_{\mathbf{k}\in\Lambda^*} \mathbf{V}_{\mathbf{k}}(t) \exp(\mathrm{i}\mathbf{k} \cdot \mathbf{x}), \qquad \mathbf{f}(t, \mathbf{x}) = \sum_{\mathbf{k}\in\Lambda^*} \mathbf{f}_{\mathbf{k}}(t) \exp(\mathrm{i}\mathbf{k} \cdot \mathbf{x}), \tag{2.3}$$

$$\nabla p(t, \mathbf{x}) = \mathbf{p}_0(t) + \mathrm{i} \sum_{\mathbf{k}\in\Lambda^*} p_{\mathbf{k}}(t)\mathbf{k} \exp(i\mathbf{k} \cdot \mathbf{x}),$$

where summation is taken over all vectors $\mathbf{k}$ of the reciprocal lattice Λ^*. For real functions $\mathbf{V}(t, \mathbf{x})$, $p(t, \mathbf{x})$ and $\mathbf{f}(t, \mathbf{x})$, the Fourier components $\mathbf{V}_{\mathbf{k}}, \mathbf{f}_{\mathbf{k}} \in \mathbb{C}^3$, and $p_{\mathbf{k}} \in \mathbb{C}$ satisfy the relations

$$\mathbf{V}_{-\mathbf{k}} = \overline{\mathbf{V}}_{\mathbf{k}}, \qquad p_{-\mathbf{k}} = \overline{p}_{\mathbf{k}}, \qquad \mathbf{f}_{-\mathbf{k}} = \overline{\mathbf{f}}_{\mathbf{k}}. \tag{2.4}$$

Vectors $\mathbf{V}_0(t)$, $\mathbf{p}_0(t)$ and $\mathbf{f}_0(t)$ are real. The incompressibility equation $\operatorname{div}\mathbf{V} = 0$ implies

$$\mathbf{k}\cdot\mathbf{V}_{\mathbf{k}} = 0. \tag{2.5}$$

Substituting formulae (2.3) into Eq. (1.1), we obtain an infinite-dimensional dynamical system for $\mathbf{n} \neq 0$

$$\dot{\mathbf{V}}_{\mathbf{n}} + \mathrm{i}\sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}}(\mathbf{V}_{\mathbf{k}}\cdot\mathbf{m})\mathbf{V}_{\mathbf{m}} + \frac{\mathrm{i}p_{\mathbf{n}}}{\rho}\mathbf{n} + \mathbf{n}^2\nu\mathbf{V}_{\mathbf{n}} - \rho^{-1}\mathbf{f}_{\mathbf{n}} = 0. \tag{2.6}$$

For $\mathbf{n} = 0$, using Eq. (2.5) we obtain

$$\dot{\mathbf{V}}_0 + \rho^{-1}(\mathbf{p}_0 - \mathbf{f}_0) = 0. \tag{2.7}$$

For $\mathbf{n} \neq 0$, all functions $p_{\mathbf{n}}$ can be excluded from Eq. (2.6). Indeed, projecting Eq. (2.6) onto vector $\mathbf{n}$ and using Eq. (2.5), we obtain

$$p_{\mathbf{n}} = -\frac{\rho}{\mathbf{n}^2}\mathbf{n}\cdot\sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}}(\mathbf{V}_{\mathbf{k}}\cdot\mathbf{m})\mathbf{V}_{\mathbf{m}} - \frac{\mathrm{i}}{\mathbf{n}^2}\mathbf{n}\cdot\mathbf{f}_{\mathbf{n}}. \tag{2.8}$$

However, vector $\mathbf{p}_0(t)$ cannot be excluded from the Navier–Stokes dynamical system as it clearly follows from Eq. (2.7). This property is closely connected with the existence of a large group of symmetries of system (2.6) and (2.7), see Section 4.

Substituting formulae (2.8) into Eq. (2.6), we obtain the equivalent form:

$$\dot{\mathbf{V}}_{\mathbf{n}} = -\mathbf{n}^2\nu\mathbf{V}_{\mathbf{n}} + \frac{\mathrm{i}}{\mathbf{n}^2}\mathbf{n}\times\left(\mathbf{n}\times\sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}}(\mathbf{V}_{\mathbf{k}}\cdot\mathbf{m})\mathbf{V}_{\mathbf{m}}\right) - \frac{1}{\rho\mathbf{n}^2}\mathbf{n}\times(\mathbf{n}\times\mathbf{f}_{\mathbf{n}}),$$

$$\dot{\mathbf{V}}_0 = \rho^{-1}(\mathbf{f}_0 - \mathbf{p}_0), \tag{2.9}$$

where we use formula $\mathbf{X} - (\mathbf{n}\cdot\mathbf{X})\mathbf{n}/\mathbf{n}^2 = -\mathbf{n}\times(\mathbf{n}\times\mathbf{X})/\mathbf{n}^2$.

(II) Using the identity

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}, \tag{2.10}$$

and Eq. (2.5), we transform Eq. (2.9) into dynamical system

$$\dot{\mathbf{V}}_{\mathbf{n}} = -\mathbf{n}^2\nu\mathbf{V}_{\mathbf{n}} + \frac{\mathrm{i}}{2\mathbf{n}^2}\mathbf{n}\times\left(\mathbf{n}\times\left(\sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}}(\mathbf{k}-\mathbf{m})\times[\mathbf{V}_{\mathbf{k}}\times\mathbf{V}_{\mathbf{m}}] + \frac{2\mathrm{i}}{\rho}\mathbf{f}_{\mathbf{n}}\right)\right), \tag{2.11}$$

where all vectors $\mathbf{k}$, $\mathbf{m}$, $\mathbf{n}$ belong to the reciprocal lattice Λ^*. In view of Proposition 5 of Section 4, we assume $\mathbf{V}_0(t) = 0$ in Eq. (2.11).

The dynamical systems (2.11) for space periodic solutions with two different triples of periods $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are equivalent if and only if the corresponding lattices Λ are connected by an orthogonal transformation. The moduli space of non-equivalent systems (2.11) has dimension 6. Indeed, if the two systems (2.11) are equivalent, then their critical points $\mathbf{V_n} = 0$ have equal eigenvalues. These eigenvalues are $\lambda_\mathbf{n} = -\mathbf{n}^2\nu$, for all vectors $\mathbf{n} \in \Lambda^*$. It is evident that the set of numbers $-\mathbf{n}^2$ for $\mathbf{n} \in \Lambda^*$ defines the lattice Λ^* up to an arbitrary rotation and inversion. Since the lattice Λ^* is defined by its basis $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$, the moduli space M of non-equivalent dynamical systems (2.11) has dimension 6.

Remark 1. The multiplicities of the eigenvalues $\lambda_\mathbf{n} = -\mathbf{n}^2\nu$ are defined by the cardinality of solutions to the equation

$$\mathbf{n}^2 = \sum_{ij}^{3} n_i n_j \mathbf{k}_i \cdot \mathbf{k}_j = C, \qquad n_i \in \mathbb{Z}. \tag{2.12}$$

If the scalar products $\mathbf{k}_i \cdot \mathbf{k}_j$ are rationally independent then Eq. (2.12) has either two integral solutions (n_1, n_2, n_3) and $(-n_1, -n_2, -n_3)$ or none. Hence on the invariant submanifold defined by the constraints (2.4) and (2.5) all eigenvalues $\lambda_\mathbf{n}$ have multiplicity 4 (since the complex vectors $\mathbf{V_n}$ are orthogonal to vectors $\mathbf{n}$). Let us show however that for a dense set of matrices $K_{ij} = \mathbf{k}_i \cdot \mathbf{k}_j$ the multiplicities of eigenvalues $\lambda_\mathbf{n} = -\mathbf{n}^2\nu$ or the number of solutions to Eq. (2.12) can be arbitrarily large when $C \longrightarrow \infty$. Indeed, suppose that matrix K_{ij} is proportional to a rational matrix: $\mathbf{k}_i \cdot \mathbf{k}_j = br_{ij}/q$, where $r_{ij}, q \in \mathbb{Z}$, $b \in \mathbb{R}$. This amounts to the condition

$$\mathbf{p}_i \cdot \mathbf{p}_j = b^{-1}(2\pi)^2(R^{-1})_{ij}, \qquad R_{ij} = \frac{r_{ij}}{q}. \tag{2.13}$$

Let P_N be the set of $(2N+1)^3$ points $\mathbf{n} = n_1\mathbf{k}_1 + n_2\mathbf{k}_2 + n_3\mathbf{k}_3 \in \Lambda^*$, where $-N \le n_i \le N$. On the P_N, we have

$$b^{-1}\mathbf{n}^2 = b^{-1}\sum_{ij} n_i n_j \mathbf{k}_i \cdot \mathbf{k}_j = \frac{r}{q} \le \frac{1}{q}\sum_{ij}|r_{ij}|N^2.$$

Hence the rational-valued function $b^{-1}\mathbf{n}^2 = r/q$ has at most $\sum |r_{ij}|N^2$ distinct values on the set P_N of $(2N+1)^3$ points. Then by Dirichlet's principle there are at least $K_N = [(2N+1)^3/\sum |r_{ij}|N^2]$ points of P_N where function $b^{-1}\mathbf{n}^2$ has the same value. As $K_N \approx cN \longrightarrow \infty$ for $N \longrightarrow \infty$, Eq. (2.12) does have arbitrarily many solutions as $C \longrightarrow \infty$. Evidently this is true for a dense set of periods $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ satisfying condition (2.13).

Remark 2. The above arguments are applicable also to any vector equations that contain the Laplace operator of velocity $\mathbf{V}(t, \mathbf{x})$, for example to the vector diffusion equations and to the viscous MHD equations.

Proposition 3. *If matrix $K_{ij} = \mathbf{k}_i \cdot \mathbf{k}_j$ has the form $K_{ij} = br_{ij}/q$, where $r_{ij}, q \in \mathbb{Z}$, then the number of solutions $\mathbf{n} \in \Lambda^*$ to Eq. (2.12) which do not belong to a finite number ℓ of lines and to a finite number m of planes becomes arbitrarily large when $C \longrightarrow \infty$.*

Proof. Let P_N^* be the set P_N without points of the ℓ lines L and the m planes P. An intersection $L \cap P_N$ has at most $2N+1$ points and $P \cap P_N$ has at most $(2N+1)^2$ points. Hence the set P_N^* has at least M_N points, $M_N = (2N+1)^3 - \ell(2N+1) - m(2N+1)^2$ and the above proof applies because $K_N^* = [M_N / \sum |r_{ij}| N^2] \approx cN \longrightarrow \infty$ when $C \longrightarrow \infty$. $\square$

3. Exact solutions with crystallographic symmetries

(I) As is known [11], a crystallographic (or space) group G is generated by three basis translations $\tilde{\mathbf{x}} = \mathbf{x} + \mathbf{p}_i$ and the generators of the point group $\Gamma \subset \mathrm{O}(3)$ combined with non-primitive translations $\mathbf{t}_Q \in \mathbb{R}^3$, where $Q \in \Gamma$. The group transforms $\tilde{\mathbf{x}} = Q\mathbf{x} + \mathbf{t}_Q$ are denoted as $\langle Q|\mathbf{t}_Q\rangle$. The multiplication law is

$$\langle Q|\mathbf{t}_Q\rangle \cdot \langle R|\mathbf{t}_R\rangle = \langle QR|\mathbf{t}_{QR}\rangle, \qquad \mathbf{t}_{QR} = Q\mathbf{t}_R + \mathbf{t}_Q. \tag{3.1}$$

The group G representation in $\mathbb{R}^3$ can be chosen to satisfy the rationality condition (2.13) [11]. Any element $\langle Q|\mathbf{t}_Q\rangle \in G$ with $Q \neq 1$, $Q \neq -1$, may have a line or a plane of eigenvectors $Q\mathbf{x} = \pm\mathbf{x}$ with eigenvalues ± 1. Since any crystallographic point group Γ has at most 48 elements we have at most 46 such eigenlines and 46 eigenplanes. Let E^* be the set of points of the lattice Λ^* which do not belong to these eigenlines and eigenplanes. It is evident that the set E^* is invariant under the group Γ action and each orbit Γ_j in E^* is isomorphic to Γ. Proposition 3 yields that Eq. (2.12) has arbitrarily many solutions $\mathbf{n} \in E^*$ when $C \longrightarrow \infty$.

There are 219 non-isomorphic space groups in three dimensions. From the tables of these groups [11], one finds that 52 of them have rotational point groups $\Gamma \subset \mathrm{SO}(3)$. Among these 52 space groups, there are 23 symmorphic groups which are the semi-direct products $\Gamma \dot{\times} \mathbb{Z}^3$ and have all translations $\mathbf{t}_Q = 0$. The other 29 non-symmorphic groups are extensions of $\mathbb{Z}^3$ by the corresponding groups Γ.

There are 11 point crystallographic groups $\Gamma \subset \mathrm{SO}(3)$: the trivial one, four cyclic groups C_n, four dihedral groups D_n, $n = 2, 3, 4, 6$, the tetrahedral group T (rotations of a regular tetrahedron), and the octahedral group O (rotations of a cube). The groups have respectively 1, n, $2n$, 12 and 24 elements. The groups C_2, C_4, D_2, D_4, T and O leave invariant a lattice with orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and the groups C_3, C_6, D_3, D_6 one with basis $\mathbf{e}_1, (1/2)\mathbf{e}_1 + (1/2)\sqrt{3}\mathbf{e}_2, \mathbf{e}_3$.

(II) A vector field $\mathbf{V}(t, \mathbf{x})$ is invariant with respect to a transform $\tilde{\mathbf{x}} = Q\mathbf{x} + \mathbf{t}_Q$ if $\mathbf{V}(t, Q\mathbf{x} + \mathbf{t}_Q) = Q\mathbf{V}(t, \mathbf{x})$. Substituting this into Fourier series (2.3), we find the invariance equations

$$\mathbf{V}_{Q\mathbf{k}} = \exp(-\mathrm{i}Q\mathbf{k} \cdot \mathbf{t}_Q)Q\mathbf{V}_{\mathbf{k}}, \qquad p_{Q\mathbf{k}} = \exp(-\mathrm{i}Q\mathbf{k} \cdot \mathbf{t}_Q)p_{\mathbf{k}}. \tag{3.2}$$

Let us construct exact G-invariant solutions in the class of Beltrami fields [2–4,12,13]. In papers [2,3] the exact solutions

$$\mathbf{V}(t, \mathbf{x}) = C_1\, \mathrm{e}^{-\alpha^2 \nu t} \int\int_{S^2} [\sin(\alpha\mathbf{k} \cdot \mathbf{x})\mathbf{T}(\mathbf{k}) + \cos(\alpha\mathbf{k} \cdot \mathbf{x})\mathbf{k} \times \mathbf{T}(\mathbf{k})]\, \mathrm{d}\sigma \tag{3.3}$$

are derived along with their generalizations for the viscous MHD equations, see also [4]. Here $\mathbf{T}(\mathbf{k})$ is an arbitrary vector field tangent to the unit sphere S^2 and $\mathrm{d}\sigma$ is an arbitrary measure on S^2. The NSE solutions (3.3) satisfy also the Beltrami equation

$$\operatorname{curl}\mathbf{V} = \alpha\mathbf{V}. \tag{3.4}$$

Substituting Fourier series (2.3) into (3.4) we obtain

$$\mathbf{k}\times\mathbf{V}_{\mathbf{k}} = -\mathrm{i}\alpha\mathbf{V}_{\mathbf{k}}. \tag{3.5}$$

Cross-multiplying this equation with vector $\mathbf{k}$ and using $\mathbf{k}\cdot\mathbf{V}_{\mathbf{k}} = 0$, we find $\mathbf{k}^2 = \alpha^2$. Taking Eq. (3.5) for wave vector $Q\mathbf{k}$ and using the invariance equations (3.2) we find

$$Q\mathbf{k}\times Q\mathbf{V}_{\mathbf{k}} = (\det Q)Q(\mathbf{k}\times\mathbf{V}_{\mathbf{k}}) = -\mathrm{i}\alpha Q\mathbf{V}_{\mathbf{k}}. \tag{3.6}$$

Eqs. (3.5) and (3.6) yield the necessary condition for G-invariance: $\det Q = 1$ for all $Q \in \Gamma$. Hence the Beltrami vector fields (3.4) can be invariant only with respect to the 52 crystallographic groups G that have purely rotational point groups $\Gamma \subset \mathrm{SO}(3)$.

Solutions to Eq. (3.5) have the form $\mathbf{V}_{\mathbf{k}} = \mathbf{A}_{\mathbf{k}} + \mathrm{i}\mathbf{k}\times\mathbf{A}_{\mathbf{k}}/\alpha$ [2], where $\mathbf{A}_{\mathbf{k}}$ are any real vectors obeying the equations $\mathbf{A}_{\mathbf{k}}\cdot\mathbf{k} = 0$, $\mathbf{A}_{-\mathbf{k}} = \mathbf{A}_{\mathbf{k}}$. From Eq. (3.2), we derive the invariance condition

$$\mathbf{A}_{Q\mathbf{k}} = Q\left(\cos(Q\mathbf{k}\cdot\mathbf{t}_Q)\mathbf{A}_{\mathbf{k}} + \frac{1}{\alpha}\sin(Q\mathbf{k}\cdot\mathbf{t}_Q)\mathbf{k}\times\mathbf{A}_{\mathbf{k}}\right). \tag{3.7}$$

Proposition 3 implies that for $\alpha^2 = C \longrightarrow \infty$ there are arbitrarily many points $\mathbf{k}\in E^*$ which satisfy Eq. (2.12) $\mathbf{k}^2 = \alpha^2 = C$. The group Γ acts on the set E^* without fixed points and vectors $\mathbf{k}$ and $-\mathbf{k}$ define different orbits of Γ. Let us choose some representative vectors $\mathbf{k}$ and $-\mathbf{k}$ for the orbits Γ_j, the real vectors $\mathbf{A}_{\mathbf{k}}$ and $\mathbf{A}_{-\mathbf{k}} = \mathbf{A}_{\mathbf{k}}$ and define vectors $\mathbf{A}_{Q\mathbf{k}}$ by formula (3.7). Hence we obtain (for $\mathbf{f}(t,\mathbf{x}) = 0$) the G-invariant space periodic solutions

$$\mathbf{V}_\alpha(t,\mathbf{x}) = \mathrm{e}^{-\alpha^2\nu t}\sum_{\mathbf{k},Q} Q[\cos(Q\mathbf{k}\cdot(\mathbf{x}-\mathbf{t}_Q))\mathbf{A}_{\mathbf{k}} - \frac{1}{\alpha}\sin(Q\mathbf{k}\cdot(\mathbf{x}-\mathbf{t}_Q))\mathbf{k}\times\mathbf{A}_{\mathbf{k}}],$$

$$p_\alpha(t,\mathbf{x}) = C_1 - \frac{\rho\mathbf{V}_\alpha^2(t,\mathbf{x})}{2}. \tag{3.8}$$

Here summation is taken over all elements $Q\in\Gamma$ and over the representative vectors $\mathbf{k}$ for every orbit Γ_j that belongs to the set $S^2\cap E^*$, $\mathbf{k}^2 = \alpha^2$. For the 23 symmorphic space groups $G = \Gamma\dot{\times}\mathbb{Z}^3$ we have $\mathbf{t}_Q = 0$ and $\mathbf{A}_{Q\mathbf{k}} = Q\mathbf{A}_{\mathbf{k}}$.

Example 4.

(a) Let lattice Λ^* has basis $\mathbf{e}_1, (1/2)\mathbf{e}_1 + (1/2)\sqrt{3}\mathbf{e}_2, \mathbf{e}_3$. The 12 vectors $\pm\mathbf{e}_1\pm\mathbf{e}_3, \pm(1/2)\mathbf{e}_1 \pm (1/2)\sqrt{3}\mathbf{e}_2 \pm \mathbf{e}_3 \in \Lambda^*$ have the same norm $|\mathbf{k}| = \sqrt{2}$ and form an orbit of the dihedral group D_6 that is generated by a 60° rotation Q_1 around $\mathbf{e}_3$ and a 180° rotation Q_2 around $\mathbf{e}_1$. Hence for $\alpha = \pm\sqrt{2}$, the corresponding exact solutions (3.8) are invariant under the crystallographic group $D_6\dot{\times}\mathbb{Z}^3$.

(b) Let lattice Λ^* has orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $\mathbf{k}_1 = (1,5,6)$, $\mathbf{k}_2 = (2,3,7)$. We have $\mathbf{k}_1^2 = \mathbf{k}_2^2 = 62$. For the 24 rotations of a unit cube, the 48 vectors $Q\mathbf{k}_1$, $Q\mathbf{k}_2$ are different for all $Q \in O$. Hence for $\alpha = \pm\sqrt{62}$, the corresponding exact solutions (3.8) are invariant with respect to the symmorphic crystallographic group $O\dot{\times}\mathbb{Z}^3$.

(c) Using the inclusions of the groups $C_2 \subset C_4 \subset D_4 \subset O$, $D_2 \subset D_4$, $T \subset O$, $C_3 \subset D_3 \subset D_6$, $C_6 \subset D_6$, we define restrictions of the examples (a) and (b) onto the subgroups Γ where vectors $\mathbf{A}_{Q\mathbf{k}} = Q\mathbf{A}_{\mathbf{k}}$ (3.7) are independent on different orbits of the group Γ. Thus we obtain solutions (3.8) that are invariant under the symmorphic crystallographic groups $\Gamma\dot{\times}\mathbb{Z}^3$.

4. Symmetries of the NSE dynamical system

(I) Let $Q \in \mathrm{O}(3)$ be any element of the holohedry $H(\Lambda^*)$ (group of orthogonal transformations that preserve the lattice Λ^*), and $\mathbf{S}(t) \in \mathbb{R}^3$ be an arbitrary smooth vector function of t. The holohedry $H(\Lambda^*)$ always contains at least two elements $Q = 1$ and $Q = -1$.

The NSE dynamical system (2.6) and (2.7) for $\mathbf{f}(t, \mathbf{x}) = 0$ has an infinite-dimensional Lie group G of symmetries

$$\tilde{\mathbf{V}}_{\mathbf{k}}(t) = \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{S}(t))Q\mathbf{V}_{Q^{-1}(\mathbf{k})}(t), \qquad \tilde{\mathbf{V}}_0(t) = Q\mathbf{V}_0(t) - \dot{\mathbf{S}}(t), \tag{4.1}$$

$$\tilde{p}_{\mathbf{k}}(t) = \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{S}(t))p_{Q^{-1}(\mathbf{k})}(t), \qquad \tilde{\mathbf{p}}_0(t) = Q\mathbf{p}_0(t) + \rho\ddot{\mathbf{S}}(t). \tag{4.2}$$

The Lie group G is a semidirect product of the holohedry $H(\Lambda^*)$ and the abelian Lie group $\mathbf{A}_0$ of vector-valued functions $\mathbf{S}(t)$, $G = H(\Lambda^*)\dot{\times}\mathbf{A}_0$. The proof follows by a straightforward verification and can be obtained also from the Lie group analysis [14] of the non-periodic Navier–Stokes equations.

(II) Substituting formulae (4.1) and (4.2) into Fourier series (2.3), we arrive at the symmetries

$$\tilde{\mathbf{V}}(t, \mathbf{x}) = Q\mathbf{V}(t, Q^{-1}[\mathbf{x} + \mathbf{S}(t)]) - \dot{\mathbf{S}}(t), \tag{4.3}$$

$$\nabla\tilde{p}(t, \mathbf{x}) = Q\nabla p(t, Q^{-1}[\mathbf{x} + \mathbf{S}(t)]) + \rho\ddot{\mathbf{S}}(t).$$

A direct substitution to Eq. (1.1) proves that transforms (4.3) with any matrix $Q \in \mathrm{O}(3)$ are symmetries of the Navier–Stokes equations for the general non-periodic case. Transforms (4.1)-(4.3) have a clear physical meaning: the invariance of the NSE equations (1.1) under transforms into an accelerated frame of reference. This is a generalization of the well-known Galilean invariance of the NSE equations where $\mathbf{S}(t) = \mathbf{u}t$, $\mathbf{u} = \text{const.}$ and $\rho\ddot{\mathbf{S}}(t) = 0$. An important point is that the transforms (4.1)-(4.3) give new solutions in the standard inertial frame of reference because the form of the Navier–Stokes equations is preserved by them.

Proposition 5. *Any smooth space periodic solution to the Navier–Stokes* equations (2.6) and (2.7) *can be transformed by symmetries* (4.1) and (4.2) *into a solution with* $\tilde{\mathbf{V}}_0(t) = 0$, $\tilde{\mathbf{p}}_0(t) = 0$.

Indeed, let $\mathbf{V}_0(0) = \mathbf{U}$. We apply the transform (4.1) and (4.2) where function $\mathbf{S}(t)$ satisfies the equations $\ddot{\mathbf{S}}(t) = -\rho^{-1}\mathbf{p}_0(t)$, $\dot{\mathbf{S}}(0) = \mathbf{U}$, $\mathbf{S}(0) = 0$, and $Q = 1$. Then we get from (4.2)

$\tilde{\mathbf{p}}_0(t) = 0$. Hence Eq. (2.7) imply $\dot{\tilde{\mathbf{V}}}_0(t) = 0$, $\tilde{\mathbf{V}}_0(t) = \tilde{\mathbf{V}}_0(0)$. The second of Eq. (4.1) gives $\tilde{\mathbf{V}}_0(0) = \mathbf{V}_0(0) - \dot{\mathbf{S}}(0) = 0$. Hence it is sufficient to consider space periodic NSE solutions with $\tilde{\mathbf{V}}_0(t) = 0$, $\tilde{\mathbf{p}}_0(t) = 0$.

In view of Proposition 5, we study below only the space periodic NSE solutions with $\mathbf{V}_0(t) = 0$, $\mathbf{p}_0(t) = 0$.

(III) For $\mathbf{V}_0 = 0$ and $\mathbf{p}_0 = 0$, the NSE dynamical system (2.11) has a three-dimensional Lie group of symmetries $G_1 = H(\Lambda^*)\dot{\times}\mathbb{T}^3$ that acts by the transformations $\tilde{\mathbf{V}}_{\mathbf{k}} = \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})Q\mathbf{V}_{Q^{-1}(\mathbf{k})}$, and $\tilde{p}_{\mathbf{k}} = \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x})p_{Q^{-1}(\mathbf{k})}$. Here vector $\mathbf{x} \in \mathbb{R}^3$ is defined mod Λ and hence belongs to the torus $\mathbb{T}^3$ and orthogonal matrix $Q \in \Gamma$.

5. Exact solutions with non-interacting Fourier modes

(I) The interaction of the $\mathbf{k}$- and $\mathbf{m}$-modes is defined by the following terms in Eq. (2.11):

$$\mathbf{Z}_{\mathbf{km}} = (\mathbf{k} + \mathbf{m}) \times [(\mathbf{k} - \mathbf{m}) \times (\mathbf{V}_{\mathbf{k}} \times \mathbf{V}_{\mathbf{m}})]. \tag{5.1}$$

The $\mathbf{k}$- and $\mathbf{m}$-modes do not interact if $\mathbf{Z}_{\mathbf{km}} = 0$.

Theorem 6. *For the Navier–Stokes* equations (1.1), *the* $\mathbf{k}$-*modes of a set S do not interact pairwise if and only if one of the following four conditions are met*:

(1) *All wave vectors* $\mathbf{k} \in S$ *are parallel.*
(2) *All wave vectors* $\mathbf{k}$ *lie in one plane L and the Fourier components* $\mathbf{V}_{\mathbf{k}}$ *are orthogonal to L.*
(3) *The vectors* $\mathbf{k}$ *belong to a circumference* $\mathbf{k}\cdot\mathbf{e} = 0$, $\mathbf{k}^2 = N$ *and vectors* $\mathbf{V}_{\mathbf{k}}$ *have the form*

$$\mathbf{V}_{\mathbf{k}} = C_{\mathbf{k}}(\alpha\mathbf{e} + \mathrm{i}\beta\mathbf{k} \times \mathbf{e}), \tag{5.2}$$

where α, β *are arbitrary reals,* $C_{-\mathbf{k}} = \bar{C}_{\mathbf{k}}$.
(4) *The vectors* $\mathbf{k}$ *belong to a sphere* $\mathbf{k}^2 = N$ *and vectors* $\mathbf{V}_{\mathbf{k}}$ *satisfy the equations*

$$\mathbf{k} \times \mathbf{V}_{\mathbf{k}} = \pm\mathrm{i}\sqrt{N}\mathbf{V}_{\mathbf{k}}, \tag{5.3}$$

with the same sign for all wave vectors $\mathbf{k} \in S$.

The proof consists on an analysis of the vector equation $\mathbf{Z}_{\mathbf{km}} = 0$ (5.1) and follows the line of the proof of paper [9] for the space periodic solutions with the standard orthogonal vector periods $(\mathbf{p}_i)_j = 2\pi\delta_{ij}$.

(II) For any set S of the pairwise non-interacting Fourier modes, the Navier–Stokes dynamical system (2.11) takes the form

$$\frac{\mathrm{d}\mathbf{V}_{\mathbf{k}}}{\mathrm{d}t} = -\mathbf{k}^2\nu\mathbf{V}_{\mathbf{k}} - \frac{1}{\rho\mathbf{k}^2}\mathbf{k} \times (\mathbf{k} \times \mathbf{f}_{\mathbf{k}}). \tag{5.4}$$

These equations define solutions to the Navier–Stokes system (2.11) provided that the vector functions $\mathbf{Pf}_{\mathbf{k}}(t) = -\mathbf{k} \times (\mathbf{k} \times \mathbf{f}_{\mathbf{k}})/(\rho\mathbf{k}^2)$ belong to the same subspaces as the Fourier components $\mathbf{V_k}$ for $\mathbf{k} \in S$ and are zero for all wave vectors $\mathbf{k}$ outside of the set S. Thus for the sub-case (1) of Theorem 6 vectors $\mathbf{f_k}$ are arbitrary for $\mathbf{k} \in S$. For the sub-case (2) vectors $\mathbf{Pf_k}$ have to be orthogonal to the plane L that contains the vectors $\mathbf{k}$. For the sub-case (3) vectors $\mathbf{Pf_k}$ should have form (5.2):

$$\mathbf{Pf_k} = (p_\mathbf{k} + \mathrm{i}q_\mathbf{k})(\alpha\mathbf{e} + \mathrm{i}\beta\mathbf{k} \times \mathbf{e}) \tag{5.5}$$

for all wave vectors $\mathbf{k}$ belonging to the circumference $\mathbf{k} \cdot \mathbf{e} = 0$, $\mathbf{k}^2 = N$. For the sub-case (4) vectors $\mathbf{Pf_k}$ have to have the form (5.3):

$$\mathbf{Pf_k} = \mathbf{P_k} \mp \frac{\mathrm{i}}{\sqrt{N}}\mathbf{k} \times \mathbf{P_k}, \quad \mathbf{k} \cdot \mathbf{P_k} = 0 \tag{5.6}$$

for all wave vectors $\mathbf{k}$ belonging to the sphere $\mathbf{k}^2 = N$.

As is known all solutions to Eq. (5.4) have the form

$$\mathbf{V_k}(t) = \mathbf{V_k}(0)\exp(-\mathbf{k}^2\nu t) + \int_0^t \mathbf{Pf_k}(\tau)\exp(\mathbf{k}^2\nu(\tau - t))\,\mathrm{d}\tau. \tag{5.7}$$

For the constant vectors $\mathbf{Pf_k}$ we have

$$\mathbf{V_k}(t) = \frac{\mathbf{Pf_k}}{\mathbf{k}^2\nu} + \tilde{\mathbf{V}}_\mathbf{k}\exp(-\mathbf{k}^2\nu t), \qquad \tilde{\mathbf{V}}_\mathbf{k} = \mathbf{V_k}(0) - \frac{\mathbf{Pf_k}}{\mathbf{k}^2\nu}. \tag{5.8}$$

The formula (5.8) implies that for any $\mathbf{V}(0, \mathbf{x}), \mathbf{f}(\mathbf{x}) \in L^2(C_0)$, the corresponding space periodic solution $\mathbf{V}(t, \mathbf{x})$ has a uniformly bounded norm $|\mathbf{V}|^2 = \sum_\mathbf{k} |\mathbf{V_k}|^2$ in the Hilbert space $L^2(C_0)$.

By the definition of the projector $\mathbf{Pf_k}$ we have $\mathbf{f_k} - \mathbf{Pf_k} = \mathrm{i}F_\mathbf{k}\mathbf{k}$ with some coefficients $F_\mathbf{k}(t)$. Therefore $(\mathbf{f_k} - \rho\mathbf{Pf_k})\exp(\mathrm{i}\mathbf{k} \cdot \mathbf{x}) = \mathrm{grad}(F_\mathbf{k}\exp(\mathrm{i}\mathbf{k} \cdot \mathbf{x}))$. Hence we find for the body force in (1.1):

$$\mathbf{f} = \mathrm{grad}\, F + \rho\sum_{\mathbf{k}\in S}\mathbf{Pf_k}\exp(\mathrm{i}\mathbf{k} \cdot \mathbf{x}), \qquad F(t, \mathbf{x}) = \sum_{\mathbf{k}\in S} F_\mathbf{k}(t)\exp(\mathrm{i}\mathbf{k} \cdot \mathbf{x}). \tag{5.9}$$

The function $F(t, \mathbf{x})$ affects only the pressure $p(t, \mathbf{x})$ and does not enter the dynamical system (5.4).

(III) To each of the four sub-cases of Theorem 6 there correspond the exact space periodic solutions (5.7) and (5.8). We present below the solutions (5.8) with constant vectors $\mathbf{Pf_k}$.

Corollary 7. *For the Navier–Stokes* equations (1.1) *with a steady body force* $\mathbf{f}(\mathbf{x})$, *there exists only four classes of the space periodic solutions with pairwise non-interacting Fourier modes*:

(1) *The two families (for the sign + and −) of exact solutions* [2–4]:

$$\mathbf{V}_{N\pm}(t, \mathbf{x}) = \frac{1}{N\nu}\sum_\mathbf{k}[\mathbf{P_k}\cos(\mathbf{k} \cdot \mathbf{x}) \pm \frac{1}{\sqrt{N}}\mathbf{k} \times \mathbf{P_k}\sin(\mathbf{k} \cdot \mathbf{x})]$$
$$+ \exp(-N\nu t)\sum_\mathbf{k}[\mathbf{B_k}\cos(\mathbf{k} \cdot \mathbf{x}) \pm \frac{1}{\sqrt{N}}\mathbf{k} \times \mathbf{B_k}\sin(\mathbf{k} \cdot \mathbf{x})], \tag{5.10}$$

where the integral vectors $\mathbf{k}$ *satisfy the equation* $\mathbf{k}^2 = N$ *and arbitrary real vectors* $\mathbf{B_k}, \mathbf{P_k}$ *conform the equations* $\mathbf{B_k} \cdot \mathbf{k} = 0$, $\mathbf{P_k} \cdot \mathbf{k} = 0$. *For* $\mathbf{P_k} = 0$, *the linear subspaces of the exact solutions* (5.10) *have dimension* $r_{3A}(N)$ *that can be arbitrarily large. The pressure is* $p(t, \mathbf{x}) = C + \rho F - \rho \mathbf{V}_{N\pm}^2/2$.

(2) *The exact solutions* [4,5]:

$$\begin{aligned}\mathbf{V}_{N\mathbf{e}}(t, \mathbf{x}) = {} & \frac{\alpha}{N\nu} \sum_{\mathbf{k}} [p_{\mathbf{k}} \cos(\mathbf{k} \cdot \mathbf{x}) - q_{\mathbf{k}} \sin(\mathbf{k} \cdot \mathbf{x})]\mathbf{e} \\ & - \frac{\beta}{N\nu} \sum_{\mathbf{k}} [p_{\mathbf{k}} \sin(\mathbf{k} \cdot \mathbf{x}) + q_{\mathbf{k}} \cos(\mathbf{k} \cdot \mathbf{x})]\mathbf{k} \times \mathbf{e} \\ & + \alpha\, \mathrm{e}^{-N\nu t} \sum_{\mathbf{k}} [a_{\mathbf{k}} \cos(\mathbf{k} \cdot \mathbf{x}) - b_{\mathbf{k}} \sin(\mathbf{k} \cdot \mathbf{x})]\mathbf{e} \\ & - \beta\, \mathrm{e}^{-N\nu t} \sum_{\mathbf{k}} [a_{\mathbf{k}} \sin(\mathbf{k} \cdot \mathbf{x}) + b_{\mathbf{k}} \cos(\mathbf{k} \cdot \mathbf{x})]\mathbf{k} \times \mathbf{e},\end{aligned} \tag{5.11}$$

where the integral vectors $\mathbf{k}$ *satisfy the equations*

$$\mathbf{k} \cdot \mathbf{e} = 0, \qquad \mathbf{k}^2 = N, \tag{5.12}$$

and $\alpha, \beta, a_{\mathbf{k}}, b_{\mathbf{k}}, p_{\mathbf{k}}, q_{\mathbf{k}}$ *are arbitrary reals,* $c_{-\mathbf{k}} = c_{\mathbf{k}}$ *for* $c = a, b, p, q$. *Solutions* (5.11) *have the form*

$$\begin{aligned}& \mathbf{V}_{N\mathbf{e}} = \alpha \mathbf{U} + \beta \operatorname{curl} \mathbf{U}, \qquad \mathbf{U}(t, \mathbf{x}) = f_N(t, \mathbf{x})\mathbf{e}, \\ & f_N = \frac{1}{N\nu} \sum_{\mathbf{k}} [p_{\mathbf{k}} \cos(\mathbf{k} \cdot \mathbf{x}) - q_{\mathbf{k}} \sin(\mathbf{k} \cdot \mathbf{x})] \\ & \qquad + \mathrm{e}^{-N\nu t} \sum_{\mathbf{k}} [a_{\mathbf{k}} \cos(\mathbf{k} \cdot \mathbf{x}) - b_{\mathbf{k}} \sin(\mathbf{k} \cdot \mathbf{x})],\end{aligned} \tag{5.13}$$

$$p(t, \mathbf{x}) = C + \rho F - \tfrac{1}{2}\rho[(\beta^2 N - \alpha^2)\mathbf{e}^2 f_N^2 + \mathbf{V}_{N\mathbf{e}}^2]. \tag{5.14}$$

(3) *The convergent series defined for any vector* $\mathbf{n} \in \Lambda^*$ [1]:

$$\begin{aligned}\mathbf{V}_{\mathbf{n}}(t, \mathbf{x}) = {} & \frac{1}{L\nu} \sum_{k=1}^{\infty} \frac{1}{k^2} [\mathbf{P}_{k\mathbf{n}} \cos(k\mathbf{n} \cdot \mathbf{x}) + \mathbf{Q}_{k\mathbf{n}} \sin(k\mathbf{n} \cdot \mathbf{x})] \\ & + \sum_{k=1}^{\infty} \exp(-k^2 L\nu t) [\mathbf{A}_{k\mathbf{n}} \cos(k\mathbf{n} \cdot \mathbf{x}) + \mathbf{B}_{k\mathbf{n}} \sin(k\mathbf{n} \cdot \mathbf{x})],\end{aligned} \tag{5.15}$$

where vectors $\mathbf{C}_{k\mathbf{n}}$ *are orthogonal to the vector* $\mathbf{n}$ *for* $\mathbf{C} = \mathbf{A}, \mathbf{B}, \mathbf{P}, \mathbf{Q}$ *and* $L = \mathbf{n}^2$. *The pressure is* $p(t, \mathbf{x}) = C + \rho F(t, \mathbf{x})$.

(4) *The convergent series defined for any two non-parallel vectors* $\mathbf{n}$ *and* $\mathbf{m} \in \Lambda^*$ [1]:

$$\mathbf{V}_{\mathbf{n},\mathbf{m}}(t,\mathbf{x}) = \frac{1}{\nu}\sum_{k,\ell=-\infty}^{\infty}\frac{1}{(k\mathbf{n}+\ell\mathbf{m})^2}[p_{k\ell}\cos((k\mathbf{n}+\ell\mathbf{m})\cdot\mathbf{x}) + q_{k\ell}\sin((k\mathbf{n} + \ell\mathbf{m})\cdot\mathbf{x})]\mathbf{n}\times\mathbf{m} + \sum_{k,\ell=-\infty}^{\infty} \mathrm{e}^{-(k\mathbf{n}+\ell\mathbf{m})^2\nu t}[a_{k\ell}\cos((k\mathbf{n}+\ell\mathbf{m})\cdot\mathbf{x}) + b_{k\ell}\sin((k\mathbf{n}+\ell\mathbf{m})\cdot\mathbf{x})]\mathbf{n}\times\mathbf{m}, \tag{5.16}$$

where k, ℓ *are arbitrary integers and* $c_{k\ell}$ *are arbitrary constants for* $c = a, b, p, q$ *that define the convergent Fourier series* (5.16). *The pressure is* $p(t,\mathbf{x}) = C + \rho F(t,\mathbf{x})$.

Proof. For the sub-case (4) of Theorem 6 the formulae (5.3) (5.6) and (5.8) after substituting into the Fourier series (2.3) give the exact solutions (5.10). The solutions (5.10) satisfy the Beltrami equation

$$\operatorname{curl}\mathbf{V}_{N\pm} = \mp\sqrt{N}\mathbf{V}_{N\pm}. \tag{5.17}$$

Let $r_{3\Lambda}(N)$ be the number of integral solutions to the equation $\mathbf{k}^2 = N$, $\mathbf{k} \in \Lambda^*$. As is known [15], the number $r_{3\Lambda}(N)$ for $(\mathbf{p}_i)_j = 2\pi\delta_{ij}$ can be arbitrarily large and the admissible integers $N \neq 4^a(8k+7)$. Each pair of $\mathbf{k}$- and $(-\mathbf{k})$-modes defines a two-dimensional family of the exact solutions (5.10). Hence for $\mathbf{P_k} = 0$ the linear subspaces $S_{N\pm}$ of exact solutions (5.10) have dimension $r_{3\Lambda}(N)$. The known identity

$$(\mathbf{V}\cdot\nabla)\mathbf{V} = \operatorname{curl}\mathbf{V}\times\mathbf{V} + \operatorname{grad}\left(\frac{\mathbf{V}^2}{2}\right), \tag{5.18}$$

and Eq. (5.17) yield $(\mathbf{V}_{N\pm}\cdot\nabla)\mathbf{V}_{N\pm} = \operatorname{grad}(\mathbf{V}_{N\pm}^2/2)$. Hence Eqs. (1.1), (5.9) and (5.4) imply for the pressure $p = C + \rho F - \rho\mathbf{V}_{N\pm}^2/2$.

The sub-case (3) of Theorem 6 for $C_{\mathbf{k}} = a_{\mathbf{k}} + \mathrm{i}b_{\mathbf{k}}$ in (5.2) and for $\mathbf{Pf_k}$ of the form (5.5) gives the exact solutions (5.11) that evidently have the form (5.13). Eq. (5.12) imply the formulae $\mathbf{e}\cdot\operatorname{grad} f_N = 0$, $\Delta f_N = -Nf_N$, $\operatorname{div}\mathbf{U} = 0$, $\Delta\mathbf{U} = -N\mathbf{U}$. Therefore using the identity $\operatorname{curl}\operatorname{curl}\mathbf{V} = \operatorname{grad}\operatorname{div}\mathbf{V} - \Delta\mathbf{V}$, we obtain $\operatorname{curl}\mathbf{V}_{N\mathbf{e}} = \beta N\mathbf{U} + \alpha\operatorname{curl}\mathbf{U}$. Hence applying the identity (2.10) we get

$$\operatorname{curl}\mathbf{V}_{N\mathbf{e}}\times\mathbf{V}_{N\mathbf{e}} = (\alpha^2 - \beta^2 N)\operatorname{curl}\mathbf{U}\times\mathbf{U} = \operatorname{grad}\frac{[(\beta^2 N - \alpha^2)\mathbf{e}^2 f_N^2]}{2},$$

and the identity (5.18) yields

$$(\mathbf{V}_{N\mathbf{e}}\cdot\nabla)\mathbf{V}_{N\mathbf{e}} = \operatorname{grad}\frac{[(\beta^2 N - \alpha^2)\mathbf{e}^2 f_N^2 + \mathbf{V}_{N\mathbf{e}}^2]}{2}.$$

Therefore formula (5.14) follows from Eqs. (1.1), (5.9) and (5.4).

The exact solutions (5.8) corresponding to the sub-cases (1) and (2) of Theorem 6 take the form (5.15) and (5.16), respectively. For these solutions we have $(\mathbf{V}\cdot\nabla)\mathbf{V}=0$; hence the pressure is $p=C+\rho F$. □

Remark 8. For the simplest case of an orthogonal lattice of periods $(\mathbf{p}_i)_j = 2\pi\delta_{ij}$ and $N=1$, $\mathbf{P_k}=0$, the solutions (5.10) after the change of time $\mathrm{d}\tau/\mathrm{d}t=\exp(-N\nu t)$ turn into the ABC-flows [16,17] for the fluid streamlines:

$$\dot{x}_1 = A\sin x_3 + C\cos x_2, \qquad \dot{x}_2 = B\sin x_1 + A\cos x_3,$$
$$\dot{x}_3 = C\sin x_2 + B\cos x_1.$$

The corresponding vectors $\mathbf{k}$ and $\mathbf{B_k}$ in (5.10) are: $\mathbf{k}_1=(1,0,0)$, $\mathbf{B}_{\mathbf{k}_1}=(0,0,B)$, $\mathbf{k}_2=(0,1,0)$, $\mathbf{B}_{\mathbf{k}_2}=(C,0,0)$, $\mathbf{k}_3=(0,0,1)$, $\mathbf{B}_{\mathbf{k}_3}=(0,A,0)$ and the minus sign is chosen in (5.10). Hence the stationary space periodic solutions (5.10) ($\mathbf{P_k}=0$) for $(\mathbf{p}_i)_j=2\pi\delta_{ij}$ and an arbitrary $N\neq 4^a(8k+7)$ form an infinite family of generalizations of the ABC-flows of dimensions $r_3(N)$ that can be arbitrarily large. The solutions also satisfy the Beltrami Eq. (5.17).

Remark 9. Let the vector $\mathbf{e}$ be one of the coordinate unit orts, for example $\mathbf{e}_3$ and $(\mathbf{p}_i)_j=\delta_{ij}$. Then Eq. (5.12) read $\mathbf{k}=k_1\mathbf{e}_1+k_2\mathbf{e}_2$, $k_1^2+k_2^2=N$. The solutions (5.11) and (5.13) for $p_\mathbf{k}=q_\mathbf{k}=0$ take the form

$$\mathbf{V}_{N3}(t,\mathbf{x})=\alpha\mathbf{U}_{N3}+\beta\,\mathrm{curl}\,\mathbf{U}_{N3}, \qquad \mathbf{U}_{N3}=f_{N3}\mathbf{e}_3,$$
$$f_{N3}=\exp(-N\nu t)\sum_{\mathbf{k}}[a_\mathbf{k}\cos(k_1x_1+k_2x_2)-b_\mathbf{k}\sin(k_1x_1+k_2x_2)]. \tag{5.19}$$

The number of the vectors $\mathbf{k}$ is equal to the $r_2(N)$ that is the number of integral solutions to the equation $k_1^2+k_2^2=N$. Let the integer $N=2^k m_1 m_2$, where $m_1=\prod q^r$, $q\equiv 1 (\mathrm{mod}\,4)$ and $m_2=\prod p^s$, $p\equiv 3(\mathrm{mod}\,4)$ where p and q are prime divisors of N. By Euler's theorem [15] the number $r_2(N)$ is zero if any of s is odd. If all s are even, then Gauss theorem [15] states that $r_2(N)=4d(m_1)$, where $d(m_1)$ is the number of divisors of m_1. Therefore the number $r_2(N)$ can be arbitrarily large. In the formula (5.19), we have for each pair $\mathbf{k}$ and $-\mathbf{k}$ the two arbitrary parameters $a_\mathbf{k}$ and $b_\mathbf{k}$ and a free parameter β/α. Hence the family of exact solutions (5.19) depends on $r_2(N)+1$ parameters.

For any two distinct positive integers k_1, k_2 there are eight integral vectors $\pm k_i\mathbf{e}_1\pm k_j\mathbf{e}_2$ with the same norm. Hence $r_2(k_1^2+k_2^2)\geq 8$. For example $65=1^2+8^2=4^2+7^2$, hence the subspace S_{65z} of exact solutions (5.19) has dimension $r_2(65)+1=17$.

6. Exact space periodic solutions to the two-dimensional NSE

Proposition 10. *For the two-dimensional Navier–Stokes equations* (1.1) *in the variables* t, x_1, x_2, *the* $\mathbf{k}$*-modes of a given set* S *do not interact pairwise if and only if one of the following two conditions are met*:

(1) *All wave vectors* $\mathbf{k}\in S$ *belong to a circumference* $\mathbf{k}^2=N$.
(2) *All wave vectors* $\mathbf{k}\in S$ *belong to a straight line* $\mathbf{k}=\lambda\mathbf{n}$.

Proof. For the space periodic two-dimensional NSE (1.1), the wave vectors $\mathbf{k}$ and the Fourier components $\mathbf{V_k}$ lie in the plane $x_3 = 0$. Hence we obtain for the interaction term (5.1): $\mathbf{Z_{km}} = (\mathbf{m}^2 - \mathbf{k}^2)\mathbf{V_k} \times \mathbf{V_m}$ and the dynamical system (2.11) turns into

$$\frac{\mathrm{d}\mathbf{V_n}}{\mathrm{d}t} = -\nu\mathbf{n}^2\mathbf{V_n} - \frac{1}{\rho\mathbf{n}^2}\mathbf{n} \times (\mathbf{n} \times \mathbf{f_n}) + \frac{\mathrm{i}}{2\mathbf{n}^2}\mathbf{n} \times \sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}} (\mathbf{m}^2 - \mathbf{k}^2)\mathbf{V_k} \times \mathbf{V_m}.$$

In view of Eqs. (2.4) and (2.5) we have $\mathbf{V_n} = \mathrm{i}v_\mathbf{n}\mathbf{n} \times \mathbf{e}_3/\mathbf{n}^2$, where $v_{-\mathbf{n}} = \overline{v}_\mathbf{n}$. Let $\mathbf{n} \times \mathbf{f_n} = -\mathrm{i}\rho f_\mathbf{n}\mathbf{e}_3$, $f_{-\mathbf{n}} = \overline{f}_\mathbf{n}$. Hence the dynamical system takes the form

$$\frac{\mathrm{d}v_\mathbf{n}}{\mathrm{d}t} = -\nu\mathbf{n}^2 v_\mathbf{n} + f_\mathbf{n} + \sum_{\mathbf{k}+\mathbf{m}=\mathbf{n}} v_\mathbf{k} v_\mathbf{m} \frac{\mathbf{k}^2 - \mathbf{m}^2}{2\mathbf{k}^2\mathbf{m}^2}(\mathbf{k} \times \mathbf{m}) \cdot \mathbf{e}_3. \tag{6.1}$$

It is evident from Eq. (6.1) that the $\mathbf{k}$- and $\mathbf{m}$-modes do not interact, or $\mathbf{Z_{km}} = 0$, if either $\mathbf{k}^2 = \mathbf{m}^2$ or the wave vectors $\mathbf{k}$ and $\mathbf{m}$ are parallel. Hence if the set S contains two non-parallel wave vectors $\mathbf{k}$ and $\mathbf{m}$ then all vectors $\mathbf{q} \in S$ belong to the circumference $\mathbf{q}^2 = N$; otherwise the set S belongs to a straight line $\mathbf{k} = \lambda\mathbf{n}$. □

Proposition 11. *For the two-dimensional Navier–Stokes* equations (1.1) *with a steady body force* $\mathbf{f}(\mathbf{x})$, *there exists only two families of the space periodic solutions with pairwise non-interacting modes*:

(1) *The exact solutions*

$$\begin{aligned}\mathbf{V}_N(t, \mathbf{x}) &= \frac{1}{N\nu} \sum_{\mathbf{k}\in\Lambda^*} [p_\mathbf{k} \sin(\mathbf{k} \cdot \mathbf{x}) + q_\mathbf{k} \cos(\mathbf{k} \cdot \mathbf{x})]\mathbf{k} \times \mathbf{e} \\ &\quad + \exp(-N\nu t) \sum_{\mathbf{k}\in\Lambda^*} [a_\mathbf{k} \sin(\mathbf{k} \cdot \mathbf{x}) + b_\mathbf{k} \cos(\mathbf{k} \cdot \mathbf{x})]\mathbf{k} \times \mathbf{e},\end{aligned} \tag{6.2}$$

where the vectors $\mathbf{k} \in \Lambda^* \subset \mathbb{R}^2$ *belong to the circumference* $\mathbf{k}^2 = N$. *For* $p_\mathbf{k} = q_\mathbf{k} = 0$, *the dimension of the linear space of solutions* (6.2) *is equal to the* $r_2(N)$, *the number of solutions to the equation* $\mathbf{k}^2 = N$, *where* $\mathbf{k} \in \Lambda^*$, *and can be arbitrarily large*.

(2) *The convergent series* (5.15) *defined for any vector* $\mathbf{n} \in \Lambda^* \subset \mathbb{R}^2$.

Proof. The sub-case (1) of Proposition 10 and formula (5.8) for $\tilde{\mathbf{V}}_\mathbf{k} = (b_\mathbf{k} - \mathrm{i}a_\mathbf{k})\mathbf{k} \times \mathbf{e}$ and $\mathbf{Pf_k} = (q_\mathbf{k} - \mathrm{i}p_\mathbf{k})\mathbf{k} \times \mathbf{e}$ define the exact solutions (6.2). For each pair of $\mathbf{k}$- and $(-\mathbf{k})$-modes we have the two-dimensional space of solutions (6.2). Hence for $p_\mathbf{k} = q_\mathbf{k} = 0$ the linear space of solutions (6.2) has dimension $r_2(N)$ that can be arbitrarily large, see Remark 9 of Section 5. The solutions (6.2) are the special cases of solutions (5.11) for $\alpha = 0, \beta = -1$. Hence the pressure is defined by formula (5.14): $p(t, \mathbf{x}) = C + \rho F - \rho(Nf_N^2 + \mathbf{V}_N^2)/2$.

The sub-case (2) of Proposition 10 gives the convergent series (5.15) and $p(t, \mathbf{x}) = C + \rho F$. □

7. Conclusions

We have derived a complete classification of the space periodic NSE solutions with pairwise non-interacting Fourier modes. The classification is independent of the vector periods $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$. However the dimensions of invariant submanifolds of solutions (5.10) depend on the periods in a drastic way and can be arbitrarily large if the rationality condition (2.13) is met. There are four infinite series of invariant submanifolds for the NSE dynamical systems (2.11) on which all solutions are smooth and exist for all moments of time $t \geq 0$. The wave vectors $\mathbf{k} \in \Lambda^*$ for them belong to the following four families of sets:

(1) the spheres $S^2 \cap \Lambda^*$: $\mathbf{k}^2 = \alpha^2$,
(2) the circumferences $S^1 \cap \Lambda^* : \mathbf{k} \cdot \mathbf{e} = 0, \mathbf{k}^2 = \alpha^2$,
(3) the straight lines $L^1 \cap \Lambda^*$: $\mathbf{k} = \lambda\mathbf{n}$,
(4) the planes $P^2 \cap \Lambda^* : \mathbf{k} \cdot \mathbf{e} = 0$.

Here $\mathbf{e} \in \Lambda$, and $\mathbf{n} \in \Lambda^*$. For the 2-*D* periodic NSE, there are only two families of solutions with non-interacting Fourier modes; the corresponding wave vectors $\mathbf{k}$ form the sets (2) and (3). The direct and inverse cascades do not work for all these solutions since there is no transfer of energy through the spectrum. These results imply that the stochastization of the solutions occurs very slowly if the initial data are in a small neighbourhood of the above invariant submanifolds. Hence the rate of stochastization is not uniform in the functional space of the NSE solutions.

The completeness of the obtained classification implies that for any solution outside of the above invariant submanifolds there exists necessarily a non-zero interaction between Fourier modes. The most complex dynamics of fluid is realized for the exact space periodic solutions (5.10) that depend on all four variables t, x, y, z and generically have no geometrical symmetries.

We have shown that dynamical systems (2.11) describing space periodic solutions with different periods $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ generically are not equivalent to each other. The systems are equivalent if and only if the corresponding lattices of periods Λ are connected by an orthogonal transformation.

We have derived exact NSE solutions (3.8) that are invariant with respect to any of the 52 crystallographic groups G that have purely rotational point groups $\Gamma \subset \mathrm{SO}(3)$. Here Γ is either a cyclic C_n or a dihedral group D_n, $n = 2, 3, 4, 6$, or the tetrahedral group T or the octahedral group O. The obtained G-invariant exact solutions depend on all four variables t, x_1, x_2, x_3.

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